

On the Radius Constants for Classes of Analytic Functions

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Abstract. Radius constants for several classes of analytic functions on the unit disk are obtained. These include the radius of starlikeness of a positive order, radius of parabolic starlikeness, radius of Bernoulli lemniscate starlikeness, and radius of uniform convexity. In the main, the radius constants obtained are sharp. Conjectures on the non-sharp constants are given.

2010 Mathematics Subject Classification: 30C45, 30C80

Keywords and phrases: Starlike functions, close-to-convex, parabolic starlike, lemniscate of Bernoulli, radius of starlikeness, radius of uniform convexity.

1. Introduction

This paper studies the class \mathcal{A} of analytic functions f in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 0 = f'(0) - 1$. Let \mathcal{S} be its subclass consisting of univalent functions. For $0 \leq \alpha < 1$, let $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ be the subclasses of \mathcal{S} consisting respectively of functions starlike of order α and convex of order α . These are functions respectively characterized by $\operatorname{Re} z f'(z)/f(z) > \alpha$ and $1 + \operatorname{Re} z f''(z)/f'(z) > \alpha$. The usual classes of starlike and convex functions are denoted by $\mathcal{S}^* := \mathcal{S}^*(0)$ and $\mathcal{C} := \mathcal{C}(0)$.

The Koebe function $k(z) = z/(1-z)^2$, which maps \mathbb{D} onto the region $\mathbb{C} \setminus \{w : w \leq -1/4\}$, is starlike but not convex. However, it is known that k maps the disk $\mathbb{D}_r := \{z \in \mathbb{D} : |z| < r\}$ onto a convex domain for every $r \leq 2 - \sqrt{3}$. Indeed, every univalent function $f \in \mathcal{S}$ maps \mathbb{D}_r onto a convex region for $r \leq 2 - \sqrt{3}$ [8, Theorem 2.13, p. 44]. This number is called the radius of convexity for \mathcal{S} .

In general, for two families \mathcal{G} and \mathcal{F} of \mathcal{A} , the \mathcal{G} -radius of \mathcal{F} , denoted by $R_{\mathcal{G}}(\mathcal{F})$, is the largest number R such that $r^{-1}f(rz) \in \mathcal{G}$ for $0 < r \leq R$, and $f \in \mathcal{F}$. Whenever \mathcal{G} is characterized by possessing a geometric property P , the number R is also referred to as the radius of property P for the class \mathcal{F} .

Several other subclasses of \mathcal{A} and \mathcal{S} are also of great interest. In [13], Kaplan introduced the close-to-convex functions $f \in \mathcal{A}$ satisfying $f'(z) \neq 0$ and $\operatorname{Re}(f'(z)/g'(z)) > 0$ for some (not necessarily normalized) convex function g . In his investigation on the Bieberbach conjecture for close-to-convex functions, Reade [28] introduced the class of close-to-starlike functions. These are functions $f \in \mathcal{A}$ with $f(z) \neq 0$ in $\mathbb{D} \setminus \{0\}$ and $\operatorname{Re}(f(z)/g(z)) > 0$ for a (not necessarily normalized) starlike function g . Close-to-convex functions are known to be univalent, but close-to-starlike functions need not. There are various other studies on classes of functions in \mathcal{A} characterized by the ratio between functions f and g belonging to particular subclasses of \mathcal{A} [4–7, 9–11, 14, 16–19, 22–26, 28, 29, 32–35, 39].

Communicated by Anton Abdulbasah Kamil.

Received: June 1, 2012?; Revised: August 16, 2012?.

Radius constants have been obtained for several of these subclasses. In [18, 19], MacGregor obtained the radius of starlikeness for the class of functions $f \in \mathcal{A}$ satisfying either

$$(1.1) \quad \operatorname{Re} \left(\frac{f(z)}{g(z)} \right) > 0 \quad \text{or} \quad \left| \frac{f(z)}{g(z)} - 1 \right| < 1$$

in \mathbb{D} for some $g \in \mathcal{C}$. Ratti [24] determined the radius of starlikeness for functions f belonging to a variant class of (1.1). In [16], MacGregor found the radius of convexity for univalent functions satisfying $|f'(z) - 1| < 1$, while Ratti [25] established the radius for functions f satisfying

$$\left| \frac{f'(z)}{g'(z)} - 1 \right| < 1 \quad (z \in \mathbb{D})$$

when g belongs to certain classes of analytic functions.

This paper finds radius constants for several classes of functions $f \in \mathcal{A}$ characterized by its ratio with a certain function g . In the following section, the classes consisting of uniformly convex functions, parabolic starlike functions, and Bernoulli lemniscate starlike functions will be brought fore to attention. In the main, the real part of the involved expressions lie in the right half-plane, and so in Section 1.2, we shall gather certain results involving functions of positive real part that will be required. Section 2 contains the main results involving the radius of Bernoulli lemniscate starlikeness, radius of starlikeness of positive order, and radius of parabolic starlikeness for several classes. These include the subclasses satisfying one of the conditions: (i) $\operatorname{Re}(f(z)/g(z)) > 0$ where $\operatorname{Re}(g(z)/z) > 0$ or $\operatorname{Re}(g(z)/z) > 1/2$, (ii) $|(f(z)/g(z)) - 1| < 1$ where $\operatorname{Re}(g(z)/z) > 0$ or g is convex, and (iii) $|(f'(z)/g'(z)) - 1| < 1$ where $\operatorname{Re} g'(z) > 0$. Section 3 is devoted to finding the radius of uniform convexity for the classes $|(f'(z)/g'(z)) - 1| < 1$, and g is either univalent, starlike or convex.

1.1. Subclasses of univalent functions

This section highlights certain important subclasses of $f \in \mathcal{S}$ that will be referred to in the sequel. A function $f \in \mathcal{S}$ is *uniformly convex* if for every circular arc γ contained in \mathbb{D} with center $\zeta \in \mathbb{D}$, the image arc $f(\gamma)$ is convex. The class \mathcal{UCV} of all uniformly convex functions was introduced by Goodman [12]. In two separate papers, Rønning [30] and Ma and Minda [15] independently proved that

$$f \in \mathcal{UCV} \iff \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \left| \frac{zf''(z)}{f'(z)} \right| \quad (z \in \mathbb{D}).$$

Rønning introduced a corresponding class of starlike functions called parabolic starlike functions. These are functions $f \in \mathcal{A}$ satisfying

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \left| \frac{zf'(z)}{f(z)} - 1 \right| \quad (z \in \mathbb{D}).$$

Denote the class of such functions by \mathcal{SP} . It is evident that $f \in \mathcal{UCV}$ if and only if $zf'(z) \in \mathcal{SP}$. A recent survey on these classes can be found in [1]. The class \mathcal{SL} , introduced by Sokół and Stankiewicz [36], consists of functions $f \in \mathcal{A}$ satisfying the inequality

$$\left| \left(\frac{zf'(z)}{f(z)} \right)^2 - 1 \right| < 1 \quad (z \in \mathbb{D}).$$

Thus a function f is in the class $\mathcal{S}_{\mathcal{L}}$ if $zf'(z)/f(z)$ lies in the region bounded by the right-half of the lemniscate of Bernoulli given by $|w^2 - 1| < 1$. Results related to the class $\mathcal{S}_{\mathcal{L}}$ can be found in [2, 3, 37, 38]. Another class $\mathcal{M}(\beta)$, $\beta > 1$, consisting of functions $f \in \mathcal{A}$ satisfying

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) < \beta \quad (z \in \mathbb{D}),$$

was investigated by Uralegaddi *et al.* [40] and Owa and Srivastava [21].

1.2. On functions with positive real part

For $0 \leq \alpha < 1$, let $\mathcal{P}(\alpha)$ denote the class of functions $p(z) = 1 + c_1z + \cdots$ satisfying the inequality $\operatorname{Re}(p(z)) > \alpha$ in \mathbb{D} and write $\mathcal{P} := \mathcal{P}(0)$. This class is related to various subclasses in \mathcal{S} as will be evident in the next section. The following results for functions in $\mathcal{P}(\alpha)$ will be required in the sequel.

Lemma 1.1. [27] *Let $p \in \mathcal{P}(\alpha)$, and $|z| = r < 1$. Then*

$$\left| p(z) - \frac{1 + (1 - 2\alpha)r^2}{1 - r^2} \right| \leq \frac{2(1 - \alpha)r}{1 - r^2}.$$

Lemma 1.2. [33] *Let $p \in \mathcal{P}(\alpha)$, and $|z| = r < 1$. Then*

$$\left| \frac{zp'(z)}{p(z)} \right| \leq \frac{2r(1 - \alpha)}{(1 - r)[1 + (1 - 2\alpha)r]}.$$

Lemma 1.3. [6, Lemma 2.4] *Let $p \in \mathcal{P}(1/2)$, and $|z| = r < 1$. Then*

$$\operatorname{Re} \frac{zp'(z)}{p(z)} \geq \begin{cases} -r/(1+r), & r < 1/3, \\ -(\sqrt{2} - \sqrt{1-r^2})^2/(1-r^2), & 1/3 \leq r \leq \sqrt{8\sqrt{2}-11} \approx 0.56. \end{cases}$$

Lemma 1.4. [2] *Let $0 < a < \sqrt{2}$. If r_a is given by*

$$r_a = \begin{cases} \left(\sqrt{1-a^2} - (1-a^2) \right)^{1/2}, & 0 < a \leq 2\sqrt{2}/3 \\ \sqrt{2} - a, & 2\sqrt{2}/3 \leq a < \sqrt{2}, \end{cases}$$

then

$$\{w : |w - a| < r_a\} \subseteq \{w : |w^2 - 1| < 1\}.$$

Lemma 1.5. [34] *Let $a > 1/2$. If the number R_a satisfies*

$$R_a = \begin{cases} a - 1/2, & 1/2 < a \leq 3/2 \\ \sqrt{2a-2}, & a \geq 3/2, \end{cases}$$

then

$$\{w : |w - a| < R_a\} \subseteq \{w : |w - 1| < \operatorname{Re} w\}.$$

2. Radius Constants for Analytic Functions

Let \mathcal{F}_1 be the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$\operatorname{Re} \left(\frac{f(z)}{g(z)} \right) > 0 \quad (z \in \mathbb{D})$$

for some $g \in \mathcal{A}$ with

$$\operatorname{Re} \left(\frac{g(z)}{z} \right) > 0 \quad (z \in \mathbb{D}).$$

Ratti [24] showed that the radius of starlikeness of functions in \mathcal{F}_1 is $\sqrt{5} - 2 \approx 0.2360$ and that the radius can be improved to $1/3$ if the function g additionally satisfies $\operatorname{Re}(g(z)/z) > 1/2$.

Theorem 2.1. *For the class \mathcal{F}_1 , the following sharp radius results hold:*

(a) *the \mathcal{SL} -radius for \mathcal{F}_1 is*

$$R_{\mathcal{SL}} = \frac{\sqrt{2} - 1}{2 + \sqrt{7 - 2\sqrt{2}}} \simeq 0.10247,$$

(b) *the $\mathcal{M}(\beta)$ -radius for \mathcal{F}_1 is*

$$R_{\mathcal{M}(\beta)} = \frac{\beta - 1}{2 + \sqrt{4 + (\beta - 1)^2}},$$

(c) *the $\mathcal{S}^*(\alpha)$ -radius for \mathcal{F}_1 is*

$$R_{\mathcal{S}^*(\alpha)} = \frac{1 - \alpha}{2 + \sqrt{5 + \alpha^2 - 2\alpha}},$$

(d) *the $\mathcal{S}_{\mathcal{P}}$ -radius for \mathcal{F}_1 is*

$$R_{\mathcal{S}_{\mathcal{P}}} = R_{\mathcal{S}^*(1/2)} = \frac{1}{4 + \sqrt{17}} \simeq 0.12311.$$

Proof. (a) Let $f \in \mathcal{F}_1$ and define $p, h : \mathbb{D} \rightarrow \mathbb{C}$ by

$$p(z) = \frac{g(z)}{z} \quad \text{and} \quad h(z) = \frac{f(z)}{g(z)}.$$

Then $p, h \in \mathcal{P}$ and $f(z) = g(z)h(z) = zp(z)h(z)$. Thus

$$\frac{zf'(z)}{f(z)} = 1 + \frac{zp'(z)}{p(z)} + \frac{zh'(z)}{h(z)}.$$

Using Lemma 1.2, it follows that

$$(2.1) \quad \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{4r}{1 - r^2}, \quad (|z| = r).$$

By Lemma 1.4, the function f satisfies

$$\left| \left(\frac{zf'(z)}{f(z)} \right)^2 - 1 \right| \leq 1$$

provided

$$\frac{4r}{1 - r^2} \leq \sqrt{2} - 1,$$

or

$$(\sqrt{2}-1)r^2 + 4r + 1 - \sqrt{2} \leq 0.$$

This inequality yields $r \leq R_{\mathcal{SL}}$.

To show that $R_{\mathcal{SL}}$ is the sharp \mathcal{SL} -radius for \mathcal{F}_1 , consider the functions f_0 and g_0 defined by

$$(2.2) \quad f_0(z) = z \left(\frac{1+z}{1-z} \right)^2 \quad \text{and} \quad g_0(z) = z \left(\frac{1+z}{1-z} \right).$$

Since $\operatorname{Re}(f_0(z)/g_0(z)) = \operatorname{Re}((1+z)/(1-z)) > 0$ and $\operatorname{Re}(g_0(z)/z) > 0$, the function f_0 belongs to \mathcal{F}_1 . Now

$$\frac{zf'_0(z)}{f_0(z)} = 1 + \frac{4z}{1-z^2}.$$

For $z = \rho := R_{\mathcal{SL}}$,

$$\left| \left(\frac{zf'_0(z)}{f_0(z)} \right)^2 - 1 \right| = \left| \left(1 + \frac{4\rho}{1-\rho^2} \right)^2 - 1 \right| = 1.$$

This shows that the radius in (a) is sharp.

(b) From inequality (2.1), it follows that

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \leq 1 + \frac{4r}{1-r^2} \leq \beta$$

if

$$(1-\beta) + 4r - (1-\beta)r^2 \leq 0,$$

that is, for $r \leq R_{\mathcal{M}(\beta)}$. For the function f_0 given by (2.2),

$$\frac{zf'_0(z)}{f_0(z)} = \frac{4\rho + 1 - \rho^2}{1 - \rho^2} = \beta \quad (z = \rho := R_{\mathcal{M}(\beta)}),$$

and so the radius is sharp.

(c) Inequality (2.1) also yields

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \geq 1 - \frac{4r}{1-r^2} \geq \alpha$$

provided

$$r^2(1-\alpha) + 4r - (1-\alpha) \leq 0.$$

The last inequality holds whenever $r \leq R_{\mathcal{S}^*(\alpha)}$. The function f_0 in (2.2) gives

$$\frac{zf'_0(z)}{f_0(z)} = \frac{1+4z-z^2}{1-z^2} = \alpha$$

for $z = -\rho := -R_{\mathcal{S}^*(\alpha)}$, and this shows that the radius in (c) is sharp.

(d) In view of Lemma 1.5, the circular disk (2.1) lies completely inside the parabolic region $\{w : |w-1| < \operatorname{Re} w\}$ provided

$$\frac{4r}{1-r^2} \leq \frac{1}{2},$$

or

$$r^2 + 8r - 1 \leq 0.$$

The last inequality holds whenever $r \leq R_{\mathcal{SP}} = R_{\mathcal{S}^*(1/2)} = 1/(4 + \sqrt{17})$ for $z = -R_{\mathcal{SP}}$.

The function f_0 in (2.2) satisfies

$$\left| \frac{zf'_0(z)}{f_0(z)} - 1 \right| = \left| \frac{4z}{1-z^2} \right| = \operatorname{Re} \frac{1-z^2+4z}{1-z^2} = \operatorname{Re} \left(\frac{zf'_0(z)}{f_0(z)} \right),$$

and so the result in (d) is sharp. I

Consider next the class \mathcal{F}_2 of functions $f \in \mathcal{A}$ satisfying

$$\operatorname{Re} \left(\frac{f(z)}{g(z)} \right) > 0 \quad (z \in \mathbb{D})$$

for some function $g \in \mathcal{A}$ with

$$\operatorname{Re} \left(\frac{g(z)}{z} \right) > \frac{1}{2} \quad (z \in \mathbb{D}).$$

Theorem 2.2. *For the class \mathcal{F}_2 , the following radius results hold:*

(a) *the \mathcal{SL} -radius is*

$$R_{\mathcal{SL}} = \frac{4-2\sqrt{2}}{\sqrt{2}(\sqrt{17-4\sqrt{2}}+3)} \simeq 0.13009,$$

(b) *the $\mathcal{M}(\beta)$ -radius is*

$$R_{\mathcal{M}(\beta)} = \frac{2(\beta-1)}{3+\sqrt{9+4\beta(\beta-1)}},$$

(c) *the $\mathcal{S}^*(\alpha)$ -radius is*

$$R_{\mathcal{S}^*(\alpha)} = \frac{2(1-\alpha)}{3+\sqrt{9-4\alpha+4\alpha^2}},$$

(d) *the \mathcal{SP} -radius satisfies*

$$R_{\mathcal{SP}} \geq -3 + \sqrt{10} \simeq 0.162278.$$

The radius in (a), (b), and (c) are sharp.

Proof. (a) Let $f \in \mathcal{F}_2$, and define functions $p, h : \mathbb{D} \rightarrow \mathbb{C}$ by

$$p(z) = \frac{g(z)}{z} \quad \text{and} \quad h(z) = \frac{f(z)}{g(z)}.$$

Then $f(z) = zh(z)p(z)$ with $h \in \mathcal{P}$ and $p \in \mathcal{P}(1/2)$. Now

$$(2.3) \quad \frac{zf'(z)}{f(z)} = 1 + \frac{zh'(z)}{h(z)} + \frac{zp'(z)}{p(z)}.$$

From Lemma 1.2, it follows that

$$(2.4) \quad \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{2r}{1-r^2} + \frac{r}{1-r} = \frac{3r+r^2}{1-r^2}.$$

By Lemma 1.4, the function f satisfies

$$\left| \left(\frac{zf'(z)}{f(z)} \right)^2 - 1 \right| \leq 1$$

provided

$$\frac{3r+r^2}{1-r^2} \leq \sqrt{2}-1,$$

or

$$\sqrt{2}r^2 + 3r + 1 - \sqrt{2} \leq 0.$$

This holds whenever $r \leq R_{\mathcal{SL}}$.

This radius $R_{\mathcal{SL}}$ is the sharp \mathcal{SL} -radius for \mathcal{F}_2 . For this purpose, let f_0 and g_0 be defined by

$$(2.5) \quad f_0(z) = \frac{z(1+z)}{(1-z)^2} \quad \text{and} \quad g_0(z) = \frac{z}{1-z}.$$

Since $\operatorname{Re}(f_0(z)/g_0(z)) > 0$ and $\operatorname{Re}(g_0(z)/z) > 1/2$, the function $f_0 \in \mathcal{F}_2$. Also

$$\frac{zf'_0(z)}{f_0(z)} = 1 + \frac{z(3+z)}{1-z^2}.$$

Thus at $z = \rho := R_{\mathcal{SL}}$,

$$\left| \left(\frac{zf'_0(z)}{f_0(z)} \right)^2 - 1 \right| = \left| \left(\frac{1+3z}{1-z^2} \right)^2 - 1 \right| = 1.$$

(b) From inequality (2.4), it follows that

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \leq \frac{3r+1}{1-r^2} \leq \beta$$

provided

$$(1+\beta)r^2 + 3r + 1 - \beta \leq 0,$$

that is, if $r \leq \mathcal{M}(\beta)$. For f_0 given by (2.5),

$$\frac{zf'_0(z)}{f_0(z)} = \frac{1+3\rho}{1-\rho^2} = \beta \quad (z = \rho := R_{\mathcal{M}(\beta)}),$$

and so the result in (b) is sharp.

(c) Using Lemmas 1.2, 1.3 and (2.3), it follows that

$$(2.6) \quad \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > 1 - \frac{2r}{1-r^2} - \frac{r}{1+r} = \frac{1-3r}{1-r^2} \geq \alpha$$

if

$$\alpha - 1 + 3r - \alpha r^2 \leq 0.$$

The last inequality holds whenever $r \leq R_{\mathcal{S}^*(\alpha)}$. For f_0 given by (2.5),

$$\frac{zf'_0(z)}{f_0(z)} = \frac{1+3z}{1-z^2} = \frac{1-3\rho}{1-\rho^2} = \alpha$$

at $z = -\rho := -R_{\mathcal{S}^*(\alpha)}$, and this shows that the result in (c) is sharp.

(d) From (2.4) and (2.6), it follows that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right)$$

if

$$\frac{1-3r}{1-r^2} \geq \frac{3r+r^2}{1-r^2},$$

that is

$$r^2 + 6r - 1 \leq 0.$$

The last inequality holds whenever $r \leq R_{\mathcal{S}\mathcal{P}}$.

Conjecture 2.1. *The sharp $\mathcal{S}\mathcal{P}$ -radius for \mathcal{F}_2 is*

$$0.162278 \simeq -3 + \sqrt{10} \leq R_{\mathcal{S}^*(1/2)} = R_{\mathcal{S}\mathcal{P}} = 3 - 2\sqrt{2} \simeq 0.171573.$$

Let \mathcal{F}_3 be the class of all functions $f \in \mathcal{A}$ satisfying the inequality

$$\left| \frac{f(z)}{g(z)} - 1 \right| < 1 \quad (z \in \mathbb{D})$$

for some function $g \in \mathcal{A}$ with

$$\operatorname{Re} \left(\frac{g(z)}{z} \right) > 0 \quad (z \in \mathbb{D}).$$

Theorem 2.3. *For the class \mathcal{F}_3 , the following radius results hold:*

(a) *the $\mathcal{S}\mathcal{L}$ -radius is*

$$R_{\mathcal{S}\mathcal{L}} = \frac{4 - 2\sqrt{2}}{\sqrt{2}(\sqrt{17 - 4\sqrt{2}} + 3)} \simeq 0.13009,$$

(b) *the $\mathcal{M}(\beta)$ -radius is*

$$R_{\mathcal{M}(\beta)} = \frac{2(\beta - 1)}{3 + \sqrt{9 + 4\beta(\beta - 1)}},$$

(c) *the $\mathcal{S}^*(\alpha)$ -radius is*

$$R_{\mathcal{S}^*(\alpha)} = \frac{2(1 - \alpha)}{3 + \sqrt{9 + 4(2 - \alpha)(1 - \alpha)}},$$

(d) *the $\mathcal{S}\mathcal{P}$ -radius is*

$$R_{\mathcal{S}\mathcal{P}} = R_{\mathcal{S}^*(1/2)} = \frac{2\sqrt{3} - 3}{3} \simeq 0.154701.$$

The radii in (c) and (d) are sharp.

Proof. (a) Let $f \in \mathcal{F}_3$. Then $|f(z)/g(z) - 1| < 1$ if and only if $\operatorname{Re}\{g(z)/f(z)\} > 1/2$ ($z \in \mathbb{D}$).

Define the functions $p, h : \mathbb{D} \rightarrow \mathbb{C}$ by

$$p(z) = \frac{g(z)}{z} \quad \text{and} \quad h(z) = \frac{g(z)}{f(z)}.$$

Then $p \in \mathcal{P}$ and $h \in \mathcal{P}(1/2)$. Now

$$\frac{zf'(z)}{f(z)} = 1 + \frac{zp'(z)}{p(z)} - \frac{zh'(z)}{h(z)},$$

and Lemma 1.2 yields

$$(2.7) \quad \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{r(3+r)}{1-r^2}.$$

By Lemma 1.4, the function f satisfies

$$\left| \left(\frac{zf'(z)}{f(z)} \right)^2 - 1 \right| \leq 1$$

provided

$$\frac{3r+r^2}{1-r^2} \leq \sqrt{2}-1,$$

or

$$\sqrt{2}r^2 + 3r + 1 - \sqrt{2} \leq 0.$$

Solving this inequality leads to $r \leq R_{\mathcal{S}\mathcal{P}}$.

(b) From inequality (2.7), it follows that

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \leq \frac{3r+1}{1-r^2} \leq \beta$$

if

$$\beta r^2 + 3r + 1 - \beta \leq 0,$$

or whenever $r \leq R_{\mathcal{M}(\beta)}$.

(c) Inequality (2.7) also yields

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) \geq \frac{1-3r-2r^2}{1-r^2} \geq \alpha$$

if

$$(2-\alpha)r^2 + 3r + \alpha - 1 \leq 0.$$

The last inequality holds if $r \leq R_{\mathcal{S}^*(\alpha)}$.

To show that $R_{\mathcal{S}^*(\alpha)}$ is the sharp $\mathcal{S}^*(\alpha)$ -radius for \mathcal{F}_3 , consider the functions f_0 and g_0 defined by

$$(2.8) \quad f_0(z) = \frac{z(1+z)^2}{1-z} \quad \text{and} \quad g_0(z) = z \left(\frac{1+z}{1-z} \right).$$

Since $|f_0(z)/g_0(z) - 1| = |z| < 1$ and $\operatorname{Re}(g_0(z)/z) > 0$, the function $f_0 \in \mathcal{F}_3$. Also

$$\operatorname{Re} \left(\frac{zf'_0(z)}{f_0(z)} \right) = \operatorname{Re} \left(\frac{1+3z-2z^2}{1-z^2} \right) = \alpha$$

at $z = -\rho := -R_{\mathcal{S}^*(\alpha)}$, and this shows that the result in (c) is sharp.

(d) In view of Lemma 1.5, the circular disk (2.7) lies completely inside the parabolic region $\{w : |w-1| < \operatorname{Re} w\}$ if

$$\frac{r(3+r)}{1-r^2} \leq \frac{1}{2},$$

or

$$3r^2 + 6r - 1 \leq 0.$$

The last inequality holds if $r \leq R_{\mathcal{S}\mathcal{P}} = R_{\mathcal{S}^*(1/2)} = 1/(4+\sqrt{17})$ for $z = -r := -R_{\mathcal{S}\mathcal{P}}$. The function f_0 given by (2.8) satisfies

$$\left| \frac{zf'_0(z)}{f_0(z)} - 1 \right| = \left| \frac{3r+r^2}{1-r^2} \right| = \frac{1-3r-2r^2}{1-r^2} = \operatorname{Re} \left(\frac{zf'_0(z)}{f_0(z)} \right).$$

Thus the radius in (d) is sharp. ■

Let \mathcal{F}_4 be the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$\left| \frac{f'(z)}{g'(z)} - 1 \right| < 1 \quad (z \in \mathbb{D})$$

for some $g \in \mathcal{A}$ with $\operatorname{Re} g'(z) > 0$ ($z \in \mathbb{D}$). In view of Alexander's relation between $\mathcal{S}_{\mathcal{P}}$ and \mathcal{UCV} , the result below follows from Theorem 2.3.

Theorem 2.4. *For the class \mathcal{F}_4 , the following sharp radius results hold:*

(1) *the $\mathcal{C}(\alpha)$ -radius is*

$$R_{\mathcal{C}(\alpha)} = \frac{2(1-\alpha)}{3 + \sqrt{9+4(\alpha-2)(\alpha-1)}},$$

(2) *the \mathcal{UCV} -radius is*

$$R_{\mathcal{UCV}} = R_{\mathcal{C}(1/2)} = \frac{2\sqrt{3}-3}{3} \simeq 0.154701.$$

Conjecture 2.2. *The sharp \mathcal{SL} -radius and sharp $\mathcal{M}(\beta)$ -radius for the class \mathcal{F}_3 are given by*

$$R_{\mathcal{SL}} = \frac{3}{2} + \frac{3}{2\sqrt{2}} - \frac{1}{2} \sqrt{\frac{27}{2} + 7\sqrt{2}} \simeq 0.142009, \quad R_{\mathcal{M}(\beta)} = \frac{2(\beta-1)}{3 + \sqrt{9+4(\beta-1)(\beta-2)}}.$$

Let \mathcal{F}_5 be the class of all functions $f \in \mathcal{A}$ satisfying the inequality

$$\left| \frac{f(z)}{g(z)} - 1 \right| < 1 \quad (z \in \mathbb{D})$$

for some convex function $g \in \mathcal{A}$.

Theorem 2.5. *For the class \mathcal{F}_5 , the following radius results hold:*

(a) *the $\mathcal{S}^*(\alpha)$ -radius is*

$$R_{\mathcal{S}^*(\alpha)} = \frac{1-\alpha}{1 + \sqrt{2+\alpha^2-2\alpha}},$$

(b) *the $\mathcal{S}_{\mathcal{P}}$ -radius is*

$$R_{\mathcal{S}_{\mathcal{P}}} = R_{\mathcal{S}^*(1/2)} = \frac{1}{\sqrt{5}+2} \simeq 0.236068,$$

(c) *the \mathcal{SL} -radius is*

$$R_{\mathcal{SL}} = 3 - 2\sqrt{2} \simeq 0.171573,$$

(d) *the $\mathcal{M}(\beta)$ -radius is*

$$R_{\mathcal{M}(\beta)} = \frac{\beta-1}{1+\beta}.$$

Proof. (a) Let $f \in \mathcal{F}_5$. Then $h = g/f \in \mathcal{P}(1/2)$ and

$$(2.9) \quad \frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} - \frac{zh'(z)}{h(z)}.$$

Since g is convex,

$$\operatorname{Re} \left(\frac{zg'(z)}{g(z)} \right) > \frac{1}{2}.$$

It follows from Lemma 1.1 that

$$(2.10) \quad \left| \frac{zg'(z)}{g(z)} - \frac{1}{1-r^2} \right| \leq \frac{r}{1-r^2}.$$

Lemma 1.2 together with (2.9) and (2.10) gives

$$(2.11) \quad \left| \frac{zf'(z)}{f(z)} - \frac{1}{1-r^2} \right| \leq \frac{r}{1-r^2} + \frac{r}{1-r} = \frac{2r+r^2}{1-r^2}.$$

Thus

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) \geq \frac{1-2r-r^2}{1-r^2} \geq \alpha$$

provided

$$(1-\alpha)r^2 + 2r + \alpha - 1 \leq 0.$$

The last inequality holds if $r \leq R_{\mathcal{S}^*(\alpha)}$.

Sharpness of the $\mathcal{S}^*(\alpha)$ -radius for \mathcal{F}_5 can be seen by considering the functions f_0 and g_0 defined by

$$(2.12) \quad f_0(z) = z \left(\frac{1+z}{1-z} \right) \quad \text{and} \quad g_0(z) = \frac{z}{1-z}.$$

Since $|f_0(z)/g_0(z) - 1| = |z| < 1$ and g_0 is convex, the function $f_0 \in \mathcal{F}_5$. Also

$$\frac{zf'_0(z)}{f_0(z)} = \frac{1-z^2+2z}{1-z^2} = \alpha$$

at $z = -r := -R_{\mathcal{S}^*(\alpha)}$.

(b) In view of Lemma 1.5, the circular disk (2.11) lies completely inside the parabolic region $\{w : |w-1| < \operatorname{Re} w\}$ when

$$\frac{2r+r^2}{1-r^2} \leq \frac{1}{1-r^2} - \frac{1}{2},$$

or

$$r^2 + 4r - 1 \leq 0.$$

The last inequality holds if $r \leq R_{\mathcal{S}} = R_{\mathcal{S}^*(1/2)}$ for $z = -r := -R_{\mathcal{S}}$. The function f_0 given by (2.12) satisfies

$$\left| \frac{zf'_0(z)}{f_0(z)} - 1 \right| = \left| \frac{2z}{1-z^2} \right| = \frac{2r}{1-r^2} = \frac{1-r^2-2r}{1-r^2} = \operatorname{Re} \left(\frac{zf'_0(z)}{f_0(z)} \right),$$

and so the radius in (b) is sharp.

(c) By Lemma 1.4 and (2.11), the function f satisfies

$$\left| \left(\frac{zf'(z)}{f(z)} \right)^2 - 1 \right| \leq 1$$

provided

$$\frac{2r+r^2}{1-r^2} \leq \sqrt{2} - \frac{1}{1-r^2},$$

or

$$(\sqrt{2}+1)r^2 + 2r - \sqrt{2} + 1 \leq 0.$$

Solving this inequality yields $r \leq R_{\mathcal{S}\mathcal{L}}$.

(d) From inequality (2.11), it follows that

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \leq \frac{2r+r^2+1}{1-r^2} \leq \beta$$

if

$$(1+\beta)r^2+2r+1-\beta \leq 0,$$

or if $r \leq R_{\mathcal{M}(\beta)}$. ■

Conjecture 2.3. *The sharp \mathcal{SL} -radius and $\mathcal{M}(\beta)$ -radius for the class \mathcal{F}_5 are given by*

$$R_{\mathcal{SL}} = -1 - \sqrt{2} + \sqrt{2(2 + \sqrt{2})} \simeq 0.198912, \quad R_{\mathcal{M}(\beta)} = \frac{(\beta - 1)}{1 + \sqrt{\beta^2 + 2 - 2\beta}}.$$

3. Radius of Uniform Convexity

This section considers sharp radius results for classes of functions introduced by Ratti [26]. Let \mathcal{F}_6 be the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$\left| \frac{f'(z)}{g'(z)} - 1 \right| < 1 \quad (z \in \mathbb{D})$$

for some univalent function $g \in \mathcal{A}$.

Theorem 3.1. *For the class \mathcal{F}_6 , the following sharp radius results hold:*

(a) *the $\mathcal{C}(\alpha)$ -radius is*

$$R_{\mathcal{C}(\alpha)} = \frac{2(1-\alpha)}{5 + \sqrt{25 + 4\alpha(\alpha-1)}},$$

(b) *the \mathcal{UCV} -radius is*

$$R_{\mathcal{UCV}} = R_{\mathcal{C}(1/2)} = 5 - 2\sqrt{6} \simeq 0.101021.$$

Proof. (a) Let $f \in \mathcal{F}_6$, and $h: \mathbb{D} \rightarrow \mathbb{C}$ be given by

$$h(z) = \frac{g'(z)}{f'(z)}.$$

Then $h \in \mathcal{P}(1/2)$ and

$$(3.1) \quad \frac{zf''(z)}{f'(z)} = \frac{zg''(z)}{g'(z)} - \frac{zh'(z)}{h(z)}.$$

Since g is univalent, it is known [8, Theorem 2.4, p. 32] that

$$(3.2) \quad \left| \frac{zg''(z)}{g'(z)} - \frac{2r^2}{1-r^2} \right| \leq \frac{4r}{1-r^2}, \quad (|z| = r).$$

Now Lemma 1.2, (3.1) and (3.2) yields

$$(3.3) \quad \left| 1 + \frac{zf''(z)}{f'(z)} - \frac{1+r^2}{1-r^2} \right| \leq \frac{5r+r^2}{1-r^2}.$$

Thus

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) \geq \frac{1-5r}{1-r^2} \geq \alpha$$

if

$$\alpha r^2 - 5r + 1 - \alpha \geq 0.$$

The last inequality holds when $r \leq R_{\mathcal{C}}(\alpha)$.

Next consider the functions f_0 and g_0 defined by

$$(3.4) \quad f'_0(z) = \frac{(1+z)^2}{(1-z)^3} \quad \text{and} \quad g_0(z) = \frac{z}{(1-z)^2}.$$

Since $|f'_0(z)/g'_0(z) - 1| = |z| < 1$ and g_0 is univalent, the function $f_0 \in \mathcal{F}_6$. Also

$$1 + \frac{zf''_0(z)}{f'_0(z)} = \frac{1+5z}{1-z^2}.$$

At $z = -r := -R_{\mathcal{C}}(\alpha)$,

$$\operatorname{Re} \left(1 + \frac{zf''_0(z)}{f'_0(z)} \right) = \frac{1-5r}{1-r^2} = \alpha.$$

This shows that the result in (a) is sharp.

(b) In view of Lemma 1.5, the circular disk (3.3) lies completely inside the parabolic region $\{w : |w-1| < \operatorname{Re} w\}$ if

$$\frac{5r+r^2}{1-r^2} \leq \frac{1+r^2}{1-r^2} - \frac{1}{2},$$

that is, provided

$$(3.5) \quad r^2 - 10r + 1 \geq 0.$$

The last inequality holds if $r \leq R_{\mathcal{UCV}} = R_{\mathcal{C}(1/2)} = 5 - 2\sqrt{6}$ for $z = -r = -R_{\mathcal{UCV}}$. The function f_0 given by (3.4) satisfies

$$\left| \frac{zf''_0(z)}{f'_0(z)} \right| = \frac{r(5-r)}{1-r^2} = \frac{1-5r}{1-r^2} = \operatorname{Re} \left(1 + \frac{zf''_0(z)}{f'_0(z)} \right),$$

and so the radius (b) is sharp. ■

Let \mathcal{F}_7 be the class of all functions $f \in \mathcal{A}$ satisfying the inequality

$$\left| \frac{f'(z)}{g'(z)} - 1 \right| < 1 \quad (z \in \mathbb{D})$$

for some starlike function $g \in \mathcal{A}$.

Theorem 3.2. *For the class \mathcal{F}_7 , the following sharp radius results hold:*

(1) *the $\mathcal{C}(\alpha)$ -radius is*

$$R_{\mathcal{C}}(\alpha) = \frac{2(1-\alpha)}{5 + \sqrt{25 + 4\alpha(\alpha-1)}},$$

(2) *the \mathcal{UCV} -radius is*

$$R_{\mathcal{UCV}} = R_{\mathcal{C}(1/2)} = 5 - 2\sqrt{6} \simeq 0.101021.$$

Proof. Since g is starlike, it is univalent, and the result follows easily from Theorem 3.1. ■

Let \mathcal{F}_8 be the class of all functions $f \in \mathcal{A}$ satisfying the inequality

$$\left| \frac{f'(z)}{g'(z)} - 1 \right| < 1 \quad (z \in \mathbb{D})$$

for some convex function $g \in \mathcal{A}$.

Theorem 3.3. *For the class \mathcal{F}_8 , the following radius results hold:*

(a) *the $\mathcal{C}(\alpha)$ -radius is*

$$R_{\mathcal{C}(\alpha)} = \frac{2(1-\alpha)}{3 + \sqrt{9+4\alpha(\alpha-1)}}.$$

(b) *the \mathcal{UCV} -radius is*

$$R_{\mathcal{UCV}} = R_{\mathcal{C}(1/2)} = 3 - 2\sqrt{2} \simeq 0.171573.$$

The results are sharp.

Proof. (a) The function g is convex, and so is univalent. Proceeding as in the proof of Theorem 3.1, evidently

$$(3.6) \quad \left| 1 + \frac{zf''(z)}{f'(z)} - \frac{1+r^2}{1-r^2} \right| \leq \frac{3r+r^2}{1-r^2},$$

which yields

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) \geq \frac{1-3r}{1-r^2} \geq \alpha,$$

or

$$\alpha r^2 - 3r + 1 - \alpha \geq 0.$$

The last inequality holds when $r \leq R_{\mathcal{C}(\alpha)}$.

Now consider functions f_0 and g_0 defined by

$$(3.7) \quad f_0'(z) = \frac{1+z}{(1-z)^2} \quad \text{and} \quad g_0(z) = \frac{z}{1-z}.$$

Since $|f_0'(z)/g_0'(z) - 1| = |z| < 1$ and g_0 is convex, the function $f_0 \in \mathcal{F}_8$. Also

$$1 + \frac{zf_0''(z)}{f_0'(z)} = \frac{1+3z}{1-z^2}.$$

At $z = -r = -R_{\mathcal{C}(\alpha)}$, then

$$\operatorname{Re} \left(1 + \frac{zf_0''(z)}{f_0'(z)} \right) = \operatorname{Re} \frac{1+3z}{1-z^2} = \frac{1-3r}{1-r^2} = \alpha.$$

This shows that the result in (a) is sharp.

(b) In view of Lemma 1.5, the circular disk (3.6) lies completely inside the parabolic region $\{w : |w-1| < \operatorname{Re} w\}$ if

$$\frac{3r+r^2}{1-r^2} \leq \frac{1+r^2}{1-r^2} - \frac{1}{2}$$

or whenever

$$(3.8) \quad r^2 - 6r + 1 \geq 0.$$

The last inequality holds if $r \leq R_{\mathcal{UCV}} = R_{\mathcal{C}(1/2)} = 3 - 2\sqrt{2}$ for $z = -r = -R_{\mathcal{UCV}}$. The function f_0 given by (3.7) satisfies,

$$\left| \left(1 + \frac{zf_0''(z)}{f_0'(z)} \right) - 1 \right| = \frac{3r - r^2}{1 - r^2} = \frac{1 - 3r}{1 - r^2} = \operatorname{Re} \left(1 + \frac{zf_0''(z)}{f_0'(z)} \right),$$

and so the result in (b) is sharp. ■

Acknowledgment. The work presented here was supported in parts by a Research University grant from Universiti Sains Malaysia, and by a research fellowship from the Council of Scientific and Industrial Research, New Delhi.

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